Irrelevant operators in the two-dimensional Ising model

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Abstract

By using conformal-field theory, we classify the possible irrelevant operators
for the Ising model with nearest-neighbour interactions on the square and
triangular lattices. We analyse the existing results for the free energy and its
derivatives and for the correlation length, showing that they are in agreement
with the conformal-field theory predictions. Moreover, these results imply
that the nonlinear scaling field of the $T\bar{T}$ operator, where T is the energy–
momentum tensor, vanishes at the critical point. Several other peculiar
cancellations are explained in terms of a number of general conjectures. We
show that all existing results on the square and triangular lattices are consistent
with the assumption that only nonzero-spin operators are present.

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1. Introduction

The role of the irrelevant operators in the two-dimensional Ising model with nearest-neighbour
interactions has been extensively discussed in the literature. The first important result is due
to Aharony and Fisher [1] who showed, by using the exact results for the free energy and
the magnetization in infinite volume, that the first correction to the susceptibility could be
explained in terms of purely analytic corrections, i.e. without introducing any contribution
due to irrelevant operators. The conclusions of Aharony and Fisher were strengthened by
the analysis of [2], which showed that the behaviour of χ up to $O(t^4)$ was fully compatible

with the absence of irrelevant operators[5]. These results gave rise to the idea (which has never received the status of an explicit conjecture as far as we know, but which has been commonly accepted in the statistical-mechanics community) that no contribution from irrelevant operators is present in the free energy of the two-dimensional Ising model with nearest-neighbour interactions. Of course, such a statement cannot be generically correct, since the lattice Ising model shows explicit violations of rotational invariance that *must* be due to nonrotationally invariant irrelevant operators. In particular, in [4], from the analysis of the mass gap, irrelevant corrections with renormalization-group (RG) dimension $y = -2$ (respectively $y = -4$) were clearly identified on the square (respectively triangular) lattice. Of course, the question remained if these operators did contribute to the free energy.

The analysis of the susceptibility of [2] has been recently extended in [5, 6]. In [6], thanks to an impressive progress in the construction and analysis of the series expansions for the susceptibility, it was clearly shown that at least two irrelevant operators contribute to the expansion of the susceptibility for $h = 0$ near the critical point. However, while these results show without doubts the presence of irrelevant operators, they do not characterize them. In particular, the identification of these irrelevant operators with the corresponding quasiprimary fields of the Ising conformal field theory (CFT) is still an open problem. In this paper, we try to make some progress in this direction.

We shall address this problem in three steps:

(1) First, we shall discuss the CFT that describes the Ising model at the critical point. We shall list all operators that may appear as irrelevant ones in the lattice Ising model.
(2) Then, we shall compare the CFT predictions with the exact results for the free energy and for the magnetization and with the results for the susceptibility reported in [6]. We shall see that these results are in perfect agreement with the RG and CFT, but have also peculiar features that can be explained if we make some additional hypotheses. The existence in the nearest-neighbour Ising model of exact transformations that map the high-temperature phase onto the low-temperature one (duality or inversion transformations) plays here a major role, indicating that these peculiar features are strictly related to the (partial) solubility of the model.
(3) The conclusions reached in the analysis of the infinite-volume free energy and of its derivatives are further strengthened by the analysis of the mass gap (exponential correlation length) and of the finite-size scaling of the free energy and of its thermal derivatives at the critical point (we use here the results of [7–9]). Finally, we analyse the finite-size scaling of the susceptibility at the critical point, showing that the dependence on the boundary conditions is in perfect agreement with the conjectures we have made.

Since the analysis is rather involved and the reader could be lost in the technical details of the forthcoming sections, we anticipate here our main findings:

- We do not find any evidence for the presence of the leading spin-zero irrelevant operator predicted by CFT, namely $T\bar{T}$ where T is the energy–momentum tensor. This result was already anticipated in [10–12] for the two-dimensional square-lattice Ising model and in [13] for the one-dimensional Ising quantum chain. Also, on the triangular lattice we do not observe the next-to-leading spin-zero irrelevant operator that has RG dimension $y = -6$.
- As mentioned above, we find unambiguous evidence of the presence of nonzero-spin irrelevant operators in the spectrum. This is not surprising, since such operators are those

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[5] We should also mention that recently a similarly unexpected cancellation was found in the free energy on the critical isotherm $T = T_c$ [3].

that describe the lattice breaking of the rotational symmetry. What is surprising is that all results can be explained in terms of the following conjecture:

"The only irrelevant operators which appear in the two-dimensional nearest-neighbour Ising model are those due to the lattice breaking of the rotational symmetry."

In some sense, it can be considered as a renewed version of the original idea of Aharony and Fisher.

Note that this conjecture applies only to the Ising model with nearest-neighbour interactions and it is not known whether other formulations of the Ising model satisfy the same conjecture (probably they don't!). Moreover, one must in principle distinguish between different lattice types. We find that both the square-lattice and the triangular-lattice results are compatible with the conjecture, but it remains to be understood if it may also hold on other less canonical lattices, for instance for honeycomb or Kagomé lattices.

This paper is organized as follows. In section 2 we describe the model, set our notations, and report the basic results that are needed in the following analysis. In section 3 we report the CFT analysis of the model at criticality and classify the possible irrelevant operators. In section 4 we discuss the infinite-volume free energy and its derivatives with respect to h for $h = 0$. We show that the exact results and the results of [6] have properties that cannot be anticipated from CFT and RG alone. In order to explain them, we put forward four conjectures that are justified in section 4.2 on the basis of the available results. In section 4.3, on the basis of the conjectures we have made, we obtain some general predictions for the susceptibility on the triangular lattice. The extension of the results of [6] to such a lattice is very important in order to understand the validity of our conjectures. In section 5 we discuss the critical behaviour of the exponential correlation length. The analysis on the triangular lattice is particularly interesting and gives strong support to the conjecture we have presented above. In sections 6 and 7 we consider the finite-size scaling of several quantities at the critical point. We show that the existence of an inversion (duality) transformation and the general conjecture presented above explain some peculiar features of the results found in [7–9]. In section 8 we summarize the results and discuss some open problems.

2. The Ising model with nearest-neighbour interactions

The two-dimensional Ising model is defined by the partition function

$$Z = \sum_{\sigma_i = \pm 1} e^{\beta \sum_{\langle n,m \rangle} \sigma_n \sigma_m + h \sum_n \sigma_n} \tag{2.1}$$

where the spin variables σ_n are defined on the sites n of a regular lattice and take the values $\{\pm 1\}$. The model has two phases: the low-temperature one in which the $\mathbb{Z}_2$ symmetry is spontaneously broken and the high-temperature one in which the symmetry is restored. The two phases are separated by a critical point which is located at $\beta = \beta_c$.

In the following we will study several observables. We define[6] the free-energy density $F(\beta, h)$, the energy per site $E(\beta, h)$, the specific heat $C(\beta, h)$, the magnetization per site $M(\beta, h)$ and the susceptibility $\chi(\beta, h)$:

$$F(\beta, h) \equiv \lim_{N \to \infty} \frac{1}{N} \log(Z(\beta, h)) \tag{2.2}$$

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[6] Note that our definitions differ by powers of the temperature and by signs from the usual thermodynamic ones. This is irrelevant for our purposes.

$$E(\beta, h) \equiv -\frac{\partial F(\beta, h)}{\partial \beta} \tag{2.3}$$

$$C(\beta, h) \equiv \frac{\partial^2 F(\beta, h)}{\partial \beta^2} \tag{2.4}$$

$$M(\beta, h) \equiv \frac{\partial F(\beta, h)}{\partial h} \tag{2.5}$$

$$\chi(\beta, h) \equiv \frac{\partial^2 F(\beta, h)}{\partial h^2}. \tag{2.6}$$

In (2.2) N is the number of sites of a finite lattice.

2.1. The square lattice

On the square lattice

$$\beta_c = \tfrac{1}{2} \log (\sqrt{2} + 1) = 0.440\,6868 \ldots \tag{2.7}$$

and we will measure the deviations from the critical temperature in terms of the variable τ introduced in [6]:

$$\tau = \frac{1}{2} \left(\frac{1}{\sinh 2\beta} - \sinh 2\beta \right). \tag{2.8}$$

For $\beta = \beta_c$, $\tau = 0$, while $\tau > 0$ (respectively $\tau < 0$) for $\beta < \beta_c$ (respectively $\beta > \beta_c$).

We will use the exact expressions for the free-energy density and magnetization in zero field given by [14]

$$F(\tau, 0) = \tfrac{1}{2} \log(2 \cosh^2 2\beta) + F^{\text{sing}}(\tau) \tag{2.9}$$

$$M(\tau, 0) = (1 - k(\tau)^2)^{1/8} \tag{2.10}$$

where

$$F^{\text{sing}}(\tau) = \int_0^\pi \frac{d\theta}{2\pi} \log \left[1 + \left(1 - \frac{\cos^2 \theta}{1 + \tau^2} \right)^{1/2} \right] \tag{2.11}$$

$$k(\tau) = (\sqrt{1 + \tau^2} + \tau)^2. \tag{2.12}$$

In this work, the duality transformation that maps the high-temperature phase onto the low-temperature phase plays an important role. The variable τ transforms naturally under such transformation, i.e. $\tau \to -\tau$. It is easy to verify that

$$k(-\tau) = \frac{1}{k(\tau)} \tag{2.13}$$

$$F^{\text{sing}}(-\tau) = F^{\text{sing}}(\tau) \tag{2.14}$$

$$k(-\tau)^{-1/8}(-\tau)^{-1/8} M(-\tau, 0) = k(\tau)^{-1/8} \tau^{-1/8} M(\tau, 0). \tag{2.15}$$

By using the exact expressions for the free energy and the magnetization, we define two functions $a(\tau)$ and $b(\tau)$ that will play a major role below. They are defined by requiring

$$F(\tau, 0) = -Aa(\tau)^2 \log |a(\tau)| + A_0(\tau) \tag{2.16}$$

$$M(\tau, 0) = Bb(\tau)|a(\tau)|^{1/8} \tag{2.17}$$

where $a(\tau)$, $b(\tau)$ and $A_0(\tau)$ are regular functions[7] of τ, $a(\tau) \approx \tau$ for $\tau \to 0$, $b(0) = 1$ and A and B are constants. Explicitly, we find

$$a(\tau) = \tau \left(1 - \frac{3}{16}\tau^2 + \frac{137}{1536}\tau^4 + O(\tau^6) \right) \tag{2.18}$$

$$b(\tau) = k(\tau)^{1/8} \left(1 + \frac{11}{128}\tau^2 - \frac{3589}{98\,304}\tau^4 + O(\tau^6) \right) \tag{2.19}$$

and

$$A = \frac{1}{2\pi} \qquad B = 2^{1/4}. \tag{2.20}$$

Under duality,

$$a(-\tau) = -a(\tau) \qquad k(-\tau)^{-1/8}b(-\tau) = k(\tau)^{-1/8}b(\tau). \tag{2.21}$$

Although the susceptibility in zero field has not been computed exactly, its behaviour for $h = 0$, $\tau \to 0$ is quite well known. In [6] the asymptotic behaviour of χ for $h = 0$ in both phases was obtained:

$$\chi_\pm(\tau) = C^\pm |\tau|^{-7/4} k(\tau)^{1/4} \hat{F}_\pm(\tau) + B_f(\tau) \tag{2.22}$$

where $\hat{F}_\pm(\tau)$ are regular functions of τ,

$$B_f(\tau) = \sum_{q=0}^{\infty} \sum_{p=0}^{\lfloor \sqrt{q} \rfloor} b^{(p,q)} \tau^q (\log |\tau|)^p \tag{2.23}$$

and τ is defined in (2.8). Here $\chi_+(\tau)$ ($\chi_-(\tau)$) is the susceptibility in the high- (low-) temperature phase.

By a careful numerical study, Orrick *et al* [6] found two additional important properties of $\hat{F}_\pm(\tau)$. First, $\hat{F}_\pm(\tau)$ are even functions of τ. There is no rigorous proof, but we note that a similar property is satisfied by the two-point function in the large-x limit, see section 5.1. Moreover, the results of [6] can be written as

$$\hat{F}_\pm(\tau) = [a(\tau)\tau^{-1}]^{-7/4} [b(\tau)k(\tau)^{-1/8}]^2 \, G_\pm(a(\tau)) \tag{2.24}$$

where $G_\pm(z)$ are even functions of z, and $a(\tau)$ and $b(\tau)$ are defined in equations (2.16) and (2.17). Explicitly

$$G_\pm(z) = 1 - \frac{1}{384}z^4 + \left(f_\pm^{(6)} - \frac{49}{1536} \right) z^6 + O(z^8) \tag{2.25}$$

where $f_\pm^{(6)}$ are numerical coefficients reported in [6]. Note the absence of the term of order z^2, a result that will play a major role below.

2.2. The triangular lattice

On the triangular lattice

$$\beta_c = \tfrac{1}{4}\log 3 = 0.274\,6531\ldots \tag{2.26}$$

We measure the deviations from the critical temperature in terms of the variable τ defined by

$$\tau \equiv \frac{1 - 4v + v^2}{\sqrt{2}v(1-v)} \tag{2.27}$$

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[7] We will call a function *regular* if it has an expansion in integer powers of τ for $\tau \to 0$.

where $v \equiv \tanh \beta$. Under the inversion transformation that maps the high-temperature phase onto the low-temperature phase,

$$v \to v' = \left(\frac{\sqrt{1 - v + v^2} - \sqrt{v}}{(1 - v)} \right)^2 \tag{2.28}$$

it transforms simply as $\tau \to -\tau$. It is thus the analogue of variable (2.8) introduced in [6].

In zero field, the free-energy density is given by [15]

$$F(\tau, 0) = \tfrac{1}{2} \log(4 \sinh 2\beta) + F^{\text{sing}}(\tau) \tag{2.29}$$

where

$$F^{\text{sing}}(\tau) = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_1}{2\pi} \frac{\mathrm{d}\phi_2}{2\pi} \log \left[3 + \tau^2 - \cos \phi_1 - \cos \phi_2 - \cos(\phi_1 + \phi_2) \right] \tag{2.30}$$

the magnetization by (2.10), where [15]

$$k(\tau) = \frac{(1 - v)^3 (1 + v)}{4v\sqrt{v(1 - v + v^2)}}. \tag{2.31}$$

Under $\tau \to -\tau$, relations (2.13), (2.14) and (2.15) hold on the triangular lattice too.

From the expressions of the magnetization and of the free energy, we can compute the functions $a(\tau)$ and $b(\tau)$ that are defined by (2.16) and (2.17). In this case, we obtain

$$a(\tau) = \tau - \frac{\tau^3}{24} + \frac{47\tau^5}{10\,368} - \frac{161\tau^7}{248\,832} + \frac{113\,191\tau^9}{1074\,954\,240} + O(\tau^{11}) \tag{2.32}$$

$$b(\tau) = k(\tau)^{1/8} \left(1 + \frac{11\tau^2}{288} - \frac{671\tau^4}{165\,888} + \frac{10\,115\tau^6}{15\,925\,248} - \frac{31\,791\,497\tau^8}{275\,188\,285\,440} + O(\tau^{10}) \right) \tag{2.33}$$

and

$$A = \frac{1}{2\sqrt{3}\pi} \qquad B = \left(\frac{8}{3} \right)^{1/8}. \tag{2.34}$$

As in the square-lattice case, the functions $a(\tau)$ and $b(\tau)$ satisfy the duality relations (2.21).

3. Conformal field theory analysis

3.1. Primary and secondary fields

The Ising model at the critical point is described by the unitary minimal CFT with central charge $c = 1/2$ [16]. Its spectrum can be divided into three conformal families characterized by different transformation properties under the dual and $\mathbb{Z}_2$ symmetries of the model. They are the identity, spin and energy families and are commonly denoted as $[I]$, $[\sigma]$, $[\epsilon]$. Let us discuss their features in detail.

- *Primary fields*. Each family contains an operator which is called a primary field (and gives the name to the entire family). Their conformal weights are $h_I = 0$, $h_\sigma = 1/16$ and $h_\epsilon = 1/2$ respectively. Since the RG eigenvalues are related to the conformal weights by $y = 2 - h - \bar{h}$, all primary fields are relevant.
- *Secondary fields*. All the remaining operators of the three families (which are called secondary fields) are generated from the primary ones by applying the generators L_{-i} and $\bar{L}_{-i}$ of the Virasoro algebra defined by

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n+m,0}. \tag{3.1}$$

It can be shown that, by applying a generator of index k, L_{-k} or $\bar{L}_{-k}$, to a field ϕ (where $\phi = I, \epsilon, \sigma$ depending on the case) of conformal weight h_ϕ, a new operator of weight $h = h_\phi + k$ is obtained. In general, any combination of L_{-i} and $\bar{L}_{-i}$ is allowed. If we denote with n the sum of the indices of the generators of type L_{-i} and with $\bar{n}$ the sum of those of type $\bar{L}_{-i}$, the conformal weight of the resulting operator is $h_\phi + \bar{h}_\phi + n + \bar{n}$. The corresponding RG eigenvalue is $y = 2 - h_\phi - \bar{h}_\phi - n - \bar{n}$.

- *Nonzero-spin states.* The secondary fields may have nonzero spin, which is given by the difference $n - \bar{n}$. In general, one is interested in quantities that are invariant under the lattice rotation group, and thus in operators that belong to its identity representation. Since the lattice invariance group is a finite subgroup of the rotation group, in the lattice discretization of a scalar operator, operators that do not have spin zero, i.e. transform nontrivially for general rotations, may appear. The invariance group of the square lattice is the finite subgroup C_4 (cyclic group of order 4), which has four representations of 'discrete' spin 0, 1, 2, and 3. An observable that transforms as a spin-j representation under the full rotation group belongs to a representation of discrete spin $j \pmod 4$ under the action of C_4. Therefore, a lattice scalar operator is expressed as a sum of continuum operators of spin $4j$, $j \in \mathbb{N}$. Analogously, on a triangular lattice the rotation group is broken to the cyclic group of order 6, C_6. In this case, a lattice scalar operator is expressed in terms of continuum operators of spin $6j$, $j \in \mathbb{N}$.

- *Null vectors.* Some of the secondary fields disappear from the spectrum due to the null-vector conditions (see [16]). In particular, this happens for one of the two states at level 2 in the $[\sigma]$ and $[\epsilon]$ families and for the unique state at level 1 in the identity family. From each null state one can generate, by applying the Virasoro operators, a whole family of null states. Hence, at level 2 in the identity family there is only one surviving secondary field, which can be identified with the stress–energy tensor T (or $\bar{T}$). The second null vector in the σ family appears at level 3 while in the ϵ family it appears at level 4. This fact will play an important role in the following.

- *Secondary fields generated by L_{-1}.* Among all secondary fields, a particular role is played by those generated by the L_{-1} Virasoro generator. L_{-1} is the generator of translations on the lattice and as a consequence, it has zero eigenvalue on translationally invariant observables. Another way to state this result is that L_{-1} can be represented as a total derivative, and as such it gives zero if applied to an operator which is the integral of a suitable density over the lattice, i.e. a translationally invariant operator.

- *Quasiprimary operators.* A quasiprimary field $|Q\rangle$ is a secondary field which satisfies the equation

$$L_1 |Q\rangle = 0. \tag{3.2}$$

This condition eliminates all the secondary fields which are generated by L_{-1}. The quasiprimary operators are the only ones that may appear in translationally invariant quantities. These fields will play a central role in our analysis since they are the natural candidates to be irrelevant operators of the model[8].

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[8] The reason is that non-quasiprimary fields can always be rewritten (using, if needed, the null vector identities), as L_{-1} acting on some combination of Virasoro generators and thus their contribution in the perturbed action is zero. Note however that this does not mean that these states do not exist, it is only when they are integrated over the two-dimensional plane that they give a vanishing contribution. Indeed, if we look at the model in the transfer matrix framework, we see that the spectrum also contains non-quasiprimary states. This fact may have important consequence if one looks at the Ising quantum chain model (which belongs to the same universality class of the two-dimensional Ising model). In this case, it has been claimed (see for instance [13, 17]) that also non-quasiprimary fields contribute to the scaling function. For an updated review on quantum spin chains and their CFT description, see [18].

3.2. Quasiprimary states and irrelevant operators

It is easy to construct, by using (3.2), all the low-lying quasiprimary states. A simple way to obtain these states (see for instance [16]) is to count, level by level, the dimension d_n of the Verma module of each conformal family. Then it is easy to show using (3.2) that the number of quasiprimary states at a given level n of the module is given by the difference $d_n - d_{n-1}$. It is clear that, in this calculation, a central role is played by the location of the null vectors. Indeed, for the Ising model we have a (relatively) small number of quasiprimary states in the lowest levels (which in turn is the reason why our analysis can be predictive) which is due to the fact that the lowest levels of the three conformal family of the Ising model are crowded with null vectors. This does not happen for higher order minimal models. In these models, one finds instead a much richer spectrum of low-lying quasiprimary states.

Here is the list of all quasiprimary operators for the Ising model up to level 10:

- In the identity family, there is one quasiprimary state at levels 2, 4 and 6 and two quasiprimary states at levels 8 and 10.
- In the energy family, there is one quasiprimary state at levels 4, 6, 7, 8 and 9 and two quasiprimary states at level 10.
- In the $[\sigma]$ family, there is one quasiprimary state at levels 3, 5, 6, 7 and 8 and two quasiprimary states at levels 9 and 10.

For all these states, it is possible to give the exact expression in terms of the Virasoro generators (even if it becomes increasingly cumbersome as the level increases). For instance, in the identity family one finds

$$Q_2^I = L_{-2}|I\rangle \tag{3.3}$$

$$Q_4^I = \left(L_{-2}^2 - \tfrac{3}{5}L_{-4}\right)|I\rangle \tag{3.4}$$

at levels 2 and 4 respectively, where we have introduced the notation Q_n^η to denote the quasiprimary state at level n in the η family.

Let us now construct from the Q_n^η listed above the irrelevant operators which could appear in any lattice translationally invariant quantity. We list below those that have RG eigenvalue $|y| < 10$. We will classify them by their spin, since operators of different spin appear on different lattices. Spin-zero operators are relevant in all cases, spin-($4n$) operators appear on the square lattice, while spin-($6n$) operators play a role only on the triangular lattice.

The spin-zero operators are the following:

- *Identity family*. $Q_2^I \bar{Q}_2^I$ whose weight is 4 and RG eigenvalue is -2; $Q_4^I \bar{Q}_4^I$ whose weight is 8 and RG eigenvalue is -6.
- *Energy family*. $Q_4^\epsilon \bar{Q}_4^\epsilon$ whose weight is 9 and RG eigenvalue is -7.
- *Spin family*. $Q_3^\sigma \bar{Q}_3^\sigma$ whose weight is $6 + \tfrac{1}{8}$ and RG eigenvalue is $-\left(4 + \tfrac{1}{8}\right)$; $Q_5^\sigma \bar{Q}_5^\sigma$ whose weight is $10 + \tfrac{1}{8}$ and RG eigenvalue is $-\left(8 + \tfrac{1}{8}\right)$.

On the square lattice, we should consider the spin-four operators:

- *Identity family*. $Q_4^I + \bar{Q}_4^I$ whose weight is 4 and RG eigenvalue is -2; $Q_6^I \bar{Q}_2^I + Q_2^I \bar{Q}_6^I$ whose weight is 8 and RG eigenvalue is -6.
- *Energy family*. $Q_4^\epsilon + \bar{Q}_4^\epsilon$ whose weight is 5 and RG eigenvalue is -3.
- *Spin family*. $\bar{Q}_3^\sigma Q_7^\sigma + \bar{Q}_7^\sigma Q_3^\sigma$ whose weight is $10 + \tfrac{1}{8}$ and RG eigenvalue is $-\left(8 + \tfrac{1}{8}\right)$.

Also the spin eight contribute on the square lattice at this order:

- *Identity family*. $Q_8^I + \bar{Q}_8^I$ whose weight is 8 and RG eigenvalue is -6;

- *Energy family.* $Q_8^\epsilon + \bar{Q}_8^\epsilon$ whose weight is 9 and RG eigenvalue is -7;
- *Spin family.* $Q_8^\sigma + \bar{Q}_8^\sigma$ whose weight is $8 + \frac{1}{8}$ and RG eigenvalue is $-\left(6 + \frac{1}{8}\right)$.

On the triangular lattice, we should consider the spin-six operators:

- *Identity family.* $Q_6^I + \bar{Q}_6^I$ whose weight is 6 and RG eigenvalue is -4; $\bar{Q}_2^I Q_8^I + \bar{Q}_8^I Q_2^I$ whose weight is 10 and RG eigenvalue is -8;
- *Energy family.* $Q_6^\epsilon + \bar{Q}_6^\epsilon$ whose weight is 7 and RG eigenvalue is -5;
- *Spin family.* $Q_6^\sigma + \bar{Q}_6^\sigma$ whose weight is $6 + \frac{1}{8}$ and RG eigenvalue is $-\left(4 + \frac{1}{8}\right)$.

Higher-order spins contribute operators with $y \leqslant -10$. For instance, in the identity family one should consider the spin-12 operator $Q_{12}^I + \bar{Q}_{12}^I$ whose weight is 12 and RG eigenvalue is -10.

Among these operators, the most important ones are: $Q_2^I \bar{Q}_2^I$ that has spin zero and $y = -2$ and should be considered both for the square and the triangular lattice; $Q_4^I + \bar{Q}_4^I$ (with $y = -2$) and $Q_6^I + \bar{Q}_6^I$ (with $y = -4$) that are the leading operators that break rotational invariance on the square and on the triangular lattice respectively. The scalar $Q_2^I \bar{Q}_2^I$ operator can be explicitly related to the energy–momentum tensor as follows: $Q_2^I \bar{Q}_2^I = T\bar{T}$. The nonzero-spin operators have a more complicated structure (since they involve higher order secondary fields, they are actually a mixture of different operators). In order to clarify their role, we shall neglect this complication and denote them (with a slight abuse of notation) with the following combinations of the energy–momentum tensor: $Q_4^I + \bar{Q}_4^I = T^2 + \bar{T}^2$, $Q_6^I + \bar{Q}_6^I = T^3 + \bar{T}^3$. These operators will play an important role in the following discussion.

As a general remark, it is important to note that, since only even-spin operators are of interest, the dimensions y of the operators satisfy the following conditions: $y \in 2\mathbb{Z}$ for the identity family $y \in 2\mathbb{Z} + 1$ for the energy family and $y \in 2\mathbb{Z} - \frac{1}{8}$ for the spin family.

Finally, we want to discuss the role of the symmetries. On the lattice, there are two exact symmetries that will play an important role:

- $\mathbb{Z}_2$ symmetry: $(h \to -h)$. Under this transformation the operators belonging to the identity and to the energy family are even, while the operators belonging to the spin family are odd.
- Duality (inversion) symmetry for $h = 0$. This transformation maps the high-temperature phase onto the low-temperature one and with our choice of variable τ (see (2.8) and (2.27) for the square and the triangular lattice respectively) it corresponds to the mapping $\tau \to -\tau$. Under this transformation (see, e.g., appendix E of [16]) the identity operators are even, the energy operators are odd, while the $[\sigma]$-family operators do not have a well-defined behaviour.

4. Infinite-volume zero-momentum quantities for $h = 0$

In this section, using the results of section 3, we shall derive the scaling behaviour of the free energy, magnetization and susceptibility at $h = 0$ and we will compare these results with the exact expressions for $F(\tau, 0)$ and $M(\tau, 0)$ and with the results of [6] on the square lattice. We will verify that the structure of these expressions is in agreement with the RG predictions, although the complicated logarithmic dependence found in [6] requires an extension of the usual scaling expressions. Moreover, the exact results and those of [6] have additional properties that are specific to the lattice nearest-neighbour Ising model and are probably not satisfied by a generic model belonging to the Ising universality class. All these properties can be explained if we make some general conjectures: they will be presented in section 4.1.

We present a general analysis for the square and the triangular lattice. In particular, we will show that the extension of the work of [6] to the triangular lattice would provide strong support for (or rule out) our conjectures.

4.1. Renormalization-group predictions and conjectures

We wish now to derive the asymptotic behaviour of $F(\tau, 0)$, $M(\tau, 0)$ and $\chi(\tau, 0)$ by using the RG approach and the classification of the irrelevant operators presented in section 3.2. We write the free energy as [19]

$$F(\tau, h) = f_b(\tau, h) + |u_t|^{2/y_t} f_\pm \left(\left\{ \frac{u_j}{|u_t|^{y_j/y_t}} \right\} \right) + |u_t|^{2/y_t} \log |u_t| \tilde{f}_\pm \left(\left\{ \frac{u_j}{|u_t|^{y_j/y_t}} \right\} \right) \tag{4.1}$$

where $f_b(\tau, h)$ is a regular function[9] of τ and h^2, u_t and u_j are nonlinear scaling fields associated with the temperature and with all other operators having corresponding dimensions $y_t = 1$ and y_j. They include the nonlinear scaling fields associated with the magnetic field with dimension $y_h = 15/8$ and those associated with all irrelevant operators. Note the presence of the logarithmic term due to a resonance[10] between the thermal and the identity operator which is responsible for the log-type singularity in the specific heat [19]. The nonlinear scaling fields are analytic functions of τ and h that have well-defined transformation properties under $h \to -h$. Those associated with the identity and the energy family are even under the transformation, while those associated with the $[\sigma]$ family (and thus u_h too) are odd. For our purposes, we can expand

$$u_t(\tau, h) = \mu_t(\tau) + \frac{h^2}{2} \lambda_t(\tau) + O(h^4) \tag{4.2}$$

$$u_j^{\text{even}}(\tau, h) = \mu_j(\tau) + \frac{h^2}{2} \lambda_j(\tau) + O(h^4) \tag{4.3}$$

$$u_j^\sigma(\tau, h) = h v_j(\tau) + O(h^3). \tag{4.4}$$

The $\mathbb{Z}_2$-even operators belong to the identity and the energy family and thus, for $h = 0$, they have well-defined properties under duality:

$$\begin{aligned} \mu_t(-\tau) &= -\mu_t(\tau) \\ \mu_j^\epsilon(-\tau) &= -\mu_j^\epsilon(\tau) \\ \mu_j^I(-\tau) &= \mu_j^I(\tau). \end{aligned} \tag{4.5}$$

In general, we expect $\mu_j^I(0) \neq 0$, and therefore we can normalize these scaling fields by requiring $\mu_j^I(0) = 1$. On the other hand, the energy-family scaling fields—including that associated with the temperature—vanish for $\tau = 0$ and thus we normalize them by requiring $\mu_j^\epsilon(\tau) \approx \tau$. The spin-family fields are normalized by requiring $v_j(0) = 1$.

Let us now present our basic conjectures that will be justified in section 4.2 on the basis of the exact expressions for the free energy and the magnetization and of the results of [6]. Two conjectures will be presented in different forms. The analysis reported here of the infinite-volume quantities gives only evidence for the weaker versions (c1) and (d0). Evidence for (c2)

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[9] Sometimes it is assumed that the bulk free energy depends on the temperature only [20, 21]. However, this conjecture is inconsistent with the rigorous results available for χ. See [22] for a critical discussion.

[10] Since secondary fields belonging to a given family differ by integers, we expect additional multiple resonances and additional terms with higher powers of $\log |u_t|$ in equation (4.1). Such higher powers have indeed been found in the analysis of χ [6].

will be provided in section 6, and evidence for (d1)/(d2) in section 5.2. As we will discuss, the analysis of χ on the triangular lattice should be able to discriminate between (d1) and (d2).

Let us now give the list of conjectures:

(a) Consider a $[\sigma]$-family operator, and let $v_j(\tau)$ be the corresponding nonlinear scaling field for $h \to 0$, cf (4.4). Then, either $v_j(\tau) = 0$, i.e. the corresponding operator is decoupled, or

$$k(-\tau)^{-1/8} v_j(-\tau) = k(\tau)^{-1/8} v_j(\tau). \tag{4.6}$$

Such a relation should be satisfied by $v_h(\tau)$ since the corresponding operator does not decouple.

(b) The functions $f_\pm$ and $\tilde{f}_\pm$ are even functions of the nonlinear scaling fields associated with the energy family.

(c1) The functions $\tilde{f}_\pm$ depend only on the $\mathbb{Z}_2$-even scaling fields.

(c2) Stronger version of the previous one: the functions $\tilde{f}_\pm$ are constant. Such a conjecture was already made by Aharony and Fisher [1].

(d0) The nonlinear scaling field of the $T\bar{T}$ operator vanishes at the critical point: $u_{T\bar{T}}(0, 0) = 0$.

(d1) Stronger version of (d0): the operator $T\bar{T}$ decouples, i.e. $u_{T\bar{T}}(\tau, h) = 0$ for all τ and h.

(d2) Stronger version of (d1): the only irrelevant operators that appear in the Ising model are the non-rotationally invariant ones.

We remark that these conjectures (in their stronger form) are sufficient to explain the existing data, but are by no means necessary. For instance, consider the three conjectures (d). All existing square-lattice results require only (d0). Conjectures (d1) and (d2) are supported by the results on the triangular lattice that will be presented in sections 5.2 and 6. There we will show $\mu_{T\bar{T}}(\tau) = o(\tau^4)$, which provides evidence for (d1), and $\mu(0) = 0$ for the scalar operator $Q_4^I \bar{Q}_4^I$ with $y = -6$, which is our motivation for the conjecture (d2). We wish also to stress that, at least in principle, some properties may hold only on a very specific lattice type and thus the observed properties on the triangular lattice may not extend to the square-lattice case.

Let us note that in the analysis of the scaling corrections the spin of the operator will play an important role. As we already mentioned in section 3.1, all operators of spin $4j$ (respectively $6j$) appear in (4.1) on the square (respectively triangular) lattice, $j \in \mathbb{N}$. However, because of the rotational invariance of the critical theory, nonzero-spin operators contribute only at second order in the Taylor expansion of the infinite-volume free energy in powers of $u_j |u_t|^{-y_j/y_t}$.

4.2. The square lattice

Let us now use the exact results for $F(\tau, 0)$ and $M(\tau, 0)$ and the results of [6] to provide evidence for the conjectures we made in the previous section.

Setting $h = 0$ in (4.1) we see that all scaling fields associated with the $[\sigma]$ family disappear. Since the dimensions of the operators belonging to the energy and to the identity family are integers, we predict

$$F(\tau, h = 0)_\pm = f_0(\tau) + f_1(\tau) \log |\tau| \tag{4.7}$$

where $f_0(\tau)$ and $f_1(\tau)$ have a regular expansion in τ. The functions $f_0(\tau)$ and $f_1(\tau)$ can in principle depend on the phase, but from the exact solution we know that this is not the case. This implies

$$\phi(\{x_j\}) \equiv f_+\left(\{x_j\}^{I,\epsilon}; \{x_j = 0\}^\sigma\right) = f_-\left(\{x_j\}^{I,\epsilon}; \{x_j = 0\}^\sigma\right) \tag{4.8}$$

$$\tilde{\phi}(\{x_j\}) \equiv \tilde{f}_+\left(\{x_j\}^{I,\epsilon}; \{x_j = 0\}^\sigma\right) = \tilde{f}_-\left(\{x_j\}^{I,\epsilon}; \{x_j = 0\}^\sigma\right). \tag{4.9}$$

Using (2.14), we find that $f_1(\tau)$ is even in τ, a property that is certainly satisfied if the conjecture (b) is true, i.e. $\tilde{\phi}(\{x_j\})$ is an even function of the energy-family scaling fields. If this is true, the energy-family scaling fields would begin to contribute to second order.

Let us now consider the magnetization in the low-temperature phase. From (4.1) we obtain ($\tau < 0$)

$$M(\tau) = \sum_{k\in[\sigma]} |\mu_t|^{2-y_k} v_k \rho_k \left(\left\{ \mu_j \mu_t^{-y_j} \right\}^{I,\epsilon} \right) + \log|\mu_t| \sum_{k\in[\sigma]} |\mu_t|^{2-y_k} v_k \tilde{\rho}_k \left(\left\{ \mu_j \mu_t^{-y_j} \right\}^{I,\epsilon} \right)$$

(4.10)

where the functions ρ_k and $\tilde{\rho}_k$ depend only on the scaling fields of the $\mathbb{Z}_2$-even operators, and the sums are over all $[\sigma]$-family operators. Now, if y_k is the dimension of an operator belonging to the $[\sigma]$ family, $y_k = -1/8 + 2n$ where n is an integer. Therefore, we predict

$$M(\tau) = (-\tau)^{1/8} M_0(\tau) + (-\tau)^{1/8} M_1(\tau) \log(-\tau)$$

(4.11)

where $M_0(\tau)$ and $M_1(\tau)$ are regular functions of τ. Now, the exact solution gives $M_1(\tau) = 0$, a property that is satisfied if the conjecture (c1) is true. Setting $M_1(\tau) = 0$, we find a perfect agreement with the exact result.

However, the exact result satisfies an additional property: using (2.15), we have

$$k(-\tau)^{-1/8} M_0(-\tau) = M_0(\tau) k(\tau)^{-1/8}.$$

(4.12)

By using the fact that $y_j = 2n - \frac{1}{8}$ (respectively $y_j = 2n - 1$, $y_j = 2n$) for a $[\sigma]$ (respectively $[\epsilon]$, $[I]$) family operator, $n \in \mathbb{Z}$, it is easy to verify that such an equation is automatically satisfied if the conjectures (a) and (b) are true.

Let us consider the susceptibility. By differentiating (4.1) and using equations (4.8) and (4.9), we obtain

$$\chi_\pm = \frac{\partial^2 f_b}{\partial h^2}\bigg|_{h=0} + \mu_t \lambda_t [2\phi(\{x_j\}) + \tilde{\phi}(\{x_j\})] + \mu_t^2 \sum_{ik\in[\sigma]} \psi_{ik,\pm}(\{x_j\}) v_i v_k |\mu_t|^{-y_i-y_k}$$

$$+ \mu_t^2 \sum_{k\in[I],[\epsilon]} \frac{\partial\phi}{\partial x_k}(\{x_j\}) |\mu_t|^{-y_k} \left(\lambda_k - y_k \mu_k \lambda_t \mu_t^{-1} \right) + 2\mu_t \lambda_t \tilde{\phi}(\{x_j\}) \log|\mu_t|$$

$$+ \mu_t^2 \log|\mu_t| \sum_{ik\in[\sigma]} \tilde{\psi}_{ik,\pm}(\{x_j\}) v_i v_k |\mu_t|^{-y_i-y_k}$$

$$+ \mu_t^2 \log|\mu_t| \sum_{k\in[I],[\epsilon]} \frac{\partial\tilde{\phi}}{\partial x_k}(\{x_j\}) |\mu_t|^{-y_k} \left(\lambda_k - y_k \mu_k \lambda_t \mu_t^{-1} \right)$$

(4.13)

where all functions depend only on the irrelevant $\mathbb{Z}_2$-even scaling fields through $x_j = \mu_j \mu_t^{-y_j}$, ϕ and $\tilde{\phi}$ are defined in equations (4.8) and (4.9), and $\psi_{ik,\pm}$ and $\tilde{\psi}_{ik,\pm}$ are second-order derivatives of $f_\pm$ and $\tilde{f}_\pm$ with respect to the $[\sigma]$-family fields. The sums over $\mathbb{Z}_2$-even fields include only the irrelevant ones—the temperature should be excluded—while the sums over $[\sigma]$-fields include both the magnetic and the irrelevant ones. Since $y_j = -1/8 + 2n$, n integer, for $[\sigma]$ operators and y_j integer for $\mathbb{Z}_2$-even operators, this result implies the expansion

$$\chi_\pm = |\tau|^{-7/4} A_\pm(\tau) + |\tau|^{-7/4} \log|\tau| B_\pm(\tau) + C(\tau) + D(\tau) \log|\tau|$$

(4.14)

where all functions are regular and only $A_\pm$ and $B_\pm$ depend on the phase.

If we now use conjecture (c1) we obtain $\tilde{\psi}_{ik,\pm} = 0$, and therefore $B_\pm(\tau) = 0$ in agreement with the results of [6].

By comparing (4.14) with (2.22), we find $B_f(\tau) = C(\tau) + D(\tau) \log|\tau|$, so that $B_f(\tau)$ should be identical in both phases, in agreement with the results of [6]. However, we predict

only a single $\log |\tau|$, while in [6] all powers appear. This means that our scaling ansatz (4.1) is not correct: there are additional resonances that give rise to a more complicated logarithmic structure.

For $\hat{F}_{\pm}(\tau)$ we find

$$\hat{F}_{\pm}(\tau) = \frac{1}{C^{\pm}} k(\tau)^{-1/4} \tau^4 \left(\frac{\mu_t}{\tau}\right)^{2+1/4} \sum_{ik \text{ odd}} \psi_{ik,\pm}(\{x_j\}) v_i v_k \mu_t^{-y_i - y_k - 1/4}. \quad (4.15)$$

By using the conjectures (a) and (b), we can show that $\hat{F}_{\pm}(\tau)$ is even in τ, in agreement with the results of [6]. Note that the functions $\lambda_j(\tau)$ instead have no specific properties under $\tau \to -\tau$ and indeed $B_f(\tau)$ contains all powers of τ.

Let us now discuss in more detail the consequences of equations (2.24) and (2.25). First, note that the most important irrelevant operator of the $[\sigma]$ family $\left(Q_3^{\sigma} \bar{Q}_3^{\sigma}\right)$ has dimension $y = -4 - 1/8$. Since $y_h = 2 - 1/8$, it gives corrections of order τ^6. Thus, neglecting corrections of this order, we need to consider only the magnetic operator (the leading one) among the $[\sigma]$-family contributions. Second, among the $\mathbb{Z}_2$-even operators, the leading ones are $T\bar{T}$ and $T^2 + \bar{T}^2$, both with $y = -2$. However, $T^2 + \bar{T}^2$ is a spin-four operator and thus it may contribute to rotationally invariant quantities only to second order, i.e. it gives corrections of order τ^4. Therefore, the leading correction (of order τ^2) can only be due to $T\bar{T}$. Accordingly, we write

$$\tilde{\phi} = -A \left(1 + \phi_1 \mu_t^2 \mu_{T\bar{T}} + O(\tau^4)\right) \quad (4.16)$$

$$\rho_h = B \left(1 + \rho_{h1} \mu_t^2 \mu_{T\bar{T}} + O(\tau^4)\right) \quad (4.17)$$

$$\psi_{\pm,hh} = C^{\pm} \left(1 + \psi_{\pm,hh1} \mu_t^2 \mu_{T\bar{T}} + O(\tau^4)\right). \quad (4.18)$$

Then, since $\mu_{T\bar{T}}(\tau)$ is an even function of τ, we have for the functions $G_{\pm}(z)$ defined in (2.24)

$$G_{\pm} = 1 + (\psi_{\pm,hh1} - 2\rho_{h1} + \phi_1) z^2 \mu_{T\bar{T}}(0) + O(z^4). \quad (4.19)$$

By comparing with (2.25), we see that one of the following two conditions must be satisfied: either $(\psi_{\pm,hh1} - 2\rho_{h1} + \phi_1) = 0$ or $\mu_{T\bar{T}}(0) = 0$. Thus, unless a miraculous cancellation occurs, the absence of the z^2 term implies our conjecture (d0).

Equation (2.25) implies also that at least one operator contributes to order τ^4 and a different one at order τ^6. Note that it is not possible that the contribution of order τ^6 is due to the nonlinear scaling field(s) already contributing to order τ^4. Indeed, if this were the case, the contribution $O(z^6)$ in (2.25) would be independent of the phase as the term $O(z^4)$ is[11]. This result is perfectly compatible with the CFT results of section 3 that predict

(1) at order τ^4, the spin-four operator $T^2 + \bar{T}^2$ appears;
(2) at order τ^6, three operators may appear: the spin-zero operators $Q_4^I \bar{Q}_4^I$ and $Q_3^{\sigma} \bar{Q}_3^{\sigma}$, and the spin-four operator $Q_4^{\epsilon} + \bar{Q}_4^{\epsilon}$.

Note that $T^2 + \bar{T}^2$ and $Q_4^{\epsilon} + \bar{Q}_4^{\epsilon}$ have $y = -2$ and $y = -3$ respectively; however, since they have spin four, they may contribute only at second order, and therefore at $O(\tau^4)$ and $O(\tau^6)$ respectively. Finally, note that (2.25) is also in perfect agreement with the stronger conjecture (d2), that only non-rotationally invariant operators are present. In this case, we have an operator that starts contributing at order τ^4 and a second one appearing at order τ^6.

At higher orders, the situation becomes more involved. Besides the contributions of the expansion of the scaling fields appearing at lower orders, at order τ^8 one must consider the

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[11] Note that this independence does not follow from the RG expressions, since the functions ψ_+ and ψ_- are expected to be different.

fourth power of the nonlinear scaling field associated with $T^2 + \bar{T}^2$. There is also a spin-zero operator $Q_4^\epsilon \bar{Q}_4^\epsilon$ with $y = -7$. However, because of the conjecture (b), we expect this operator to contribute only to second order and therefore starting at $O(\tau^{14})$.

It is interesting to note that, if the conjecture (d0) is true, equations (2.16) and (2.17) provide the first terms of the expansion of $\mu_t(\tau)$ and $v_h(\tau)$ in powers of τ. Explicitly

$$\mu_t(\tau) = \tau \left(1 - \frac{3}{16}\tau^2 + O(\tau^4) \right) \tag{4.20}$$

$$v_h(\tau) = k(\tau)^{1/8} \left(1 + \frac{11}{128}\tau^2 + O(\tau^4) \right). \tag{4.21}$$

Such expansions already appear in [22], but assume a very simple form in the variable τ.

Finally, let us see what information we can obtain from $B_f(\tau)$. As we have already noted, our expressions are not compatible with (2.23) because of the presence of higher powers of $\log \tau$. We assume here that our parametrization of the free energy gives the correct expression of $B_f(\tau)$ up to terms of order τ^4, since at this order a $\log^2 \tau$ appears. Under this assumption, we can compute the first terms in the expansion of $\lambda_t(\tau)$. We compare the terms proportional to $\log |\tau|$, writing

$$2\mu_t(\tau)\lambda_t(\tau)\tilde{\phi}(\{0\}) = \sum_{q=1}^{3} b^{(1,q)}\tau^q + O(\tau^4). \tag{4.22}$$

Using $\tilde{\phi}(0) = -1/(2\pi)$, this gives for $\lambda_t(\tau)$

$$\lambda_t(\tau) = k(\tau)^{1/4} \sum_{k=0}^{\infty} \lambda_{tk}\tau^k \tag{4.23}$$

where

$\lambda_{t0} = -0.101\,637\,648\,975\,279\,876\,579\,045\,203\,385\,062\,636\,255\,484\,896\,85$

$\lambda_{t1} = 0$ \hfill (4.24)

$\lambda_{t2} = -0.000\,912\,698\,513\,043\,685\,863\,484\,370\,258\,366\,986\,546\,254\,622.$

It remains unclear why, by factoring out the term $k(\tau)^{1/4}$, the linear term in $\lambda_t(\tau)$ vanishes. Note that the value of λ_{t2} is correct only if the conjecture (d0) holds.

4.3. The triangular lattice

It is very interesting to extend the results of [6] to the triangular lattice. Indeed, in this case it is possible to make a much stronger test of the conjectures we have made.

First, it is easy to see that the exact results [15] for the free energy and the magnetization are fully compatible with the conjectures we have made. Then, let us derive the behaviour of the susceptibility. Equation (4.14) is lattice independent and it implies (apart from the logarithmic structure) (2.22). Therefore, the expansion on the triangular lattice should also have the form (2.22). Also, according to conjectures (a) and (b), we expect $\hat{F}(\tau)$ to be even in τ, where now τ is defined in (2.27): some evidence will be provided in section 5.2. Therefore, (2.24) should hold with $G_\pm(z)$ even in z.

Finally, we wish to predict which powers of z should be absent in the expansion of $G_\pm(z)$. This depends on the operators that can appear. CFT predicts the following:

(1) At order τ^2 we should consider $T\bar{T}$;
(2) At order τ^6 we should consider the spin-zero operators $Q_4^I \bar{Q}_4^I$ and $Q_3^\sigma \bar{Q}_3^\sigma$;

(3) At order τ^8 we should consider the spin-six operator $Q_6^I + \bar{Q}_6^I$;
(4) At order τ^{10} we should consider the spin-zero operators $Q_6^I \bar{Q}_6^I$, $Q_5^\sigma \bar{Q}_5^\sigma$ and the spin-six operators $Q_6^\epsilon + \bar{Q}_6^\epsilon$, $Q_6^\sigma + \bar{Q}_6^\sigma$.

As we have already mentioned, spin-six operators contribute to second order in rotationally invariant quantities. Moreover, we have not indicated powers of lower-order operators and the [ϵ]-family operator $Q_4^\epsilon \bar{Q}_4^\epsilon$ that, according to conjecture (b), should contribute corrections of order τ^{14}.

From this classification, we have the following possibilities:

(1) If $T\bar{T}$ is present, the term of order z^2 should be present barring miraculous cancellations.
(2) If the conjecture (d0) is true, as on the square lattice, while the conjecture (d1) is false so that $\mu_{T\bar{T}}(\tau) \sim \tau^2$, then the term of order z^2 should be absent and the term of order z^4 should be nonvanishing.
(3) If the conjecture (d1) is valid, both terms of order z^2 and z^4 should be absent;
(4) If the stronger conjecture (d2) is true, i.e. if only non-rotationally invariant operators are present, the term of order z^6 is also absent. More precisely, this cancellation would imply $\mu(0) = 0$ for $Q_4^I \bar{Q}_4^I$, $v(0) = 0$ for $Q_3^\sigma \bar{Q}_3^\sigma$, and $\mu_{T\bar{T}}(\tau) \sim o(\tau^4)$. We expect the term of order z^8 to be nonvanishing since at this order the spin-six operator $Q_6^I + \bar{Q}_6^I$ should contribute.

The triangular lattice is therefore a better testing ground for our conjectures. Indeed, conjecture (d1) requires two coefficients to vanish, a very nontrivial fact. Moreover, we are able to distinguish between conjectures (d1) and (d2).

5. The large-distance behaviour of the two-point function

In this section, we will study the large-distance behaviour of the two-point function on the square lattice, reviewing in part the results of [12], and on the triangular lattice. The square-lattice analysis will confirm the validity of conjecture (d0), i.e. $\mu_{T\bar{T}}(0) = 0$. Much more interesting is the analysis on the triangular lattice which will show that $\mu_{T\bar{T}}(\tau) = o(\tau^4)$, thus providing strong support to conjecture (d1). We will also find that the subleading corrections due to the zero-spin operator with $y = -6$ are absent, in agreement with the conjecture presented in the introduction (conjecture (d2) of section 4.1).

5.1. The square lattice

Let us now consider the large-distance behaviour of the two-point function for $h = 0$, $\tau > 0$. For large $|x|$ it has the form [23]

$$G(x, y; \tau) = Z(\tau) \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \frac{e^{ik_1 x + ik_2 y}}{\Delta_s(k) + M_s(\tau)^2} \tag{5.1}$$

where

$$\Delta_s(k) = 4 \sin^2 \frac{k_1}{2} + 4 \sin^2 \frac{k_2}{2} \tag{5.2}$$

$$Z(\tau) = \sqrt{8} \tau^{1/4} k(\tau)^{1/4} (1 + \tau^2)^{1/8} = 2(k(\tau)^2 - 1)^{1/4} \tag{5.3}$$

$$M_s(\tau)^2 = 4(\sqrt{1 + \tau^2} - 1). \tag{5.4}$$

From these expressions, we can compute the angle-dependent correlation length $\xi(\theta)$ defined from the large-distance behaviour of the two-point function along a direction forming an angle

θ with the side of the lattice. We obtain

$$\xi(\theta) = \frac{1}{\sqrt{2}a(\tau)} \left[1 + \frac{a(\tau)^2}{48} \cos 4\theta \right.$$
$$\left. + a(\tau)^4 \left(\frac{1}{3072} - \frac{1}{320} \cos 4\theta - \frac{5}{9216} \cos 8\theta \right) + O(a(\tau)^6) \right] \tag{5.5}$$

where $a(\tau)$ is defined by equations (2.16) and (2.18). As already observed in [4], this expansion shows the presence of a correction of order τ^2 due to the leading irrelevant operator breaking rotational invariance. However, the interesting additional feature is that this term is the only one, i.e. there is no correction due to the rotationally invariant subleading operators [12]. This result is naturally interpreted: the correction we find is due to the spin-four operator $T^2 + \bar{T}^2$ and there is no contribution due the scalar operator $T\bar{T}$. At order τ^4 there is a scalar term, but this does not require the presence of a scalar operator: the angle-independent contribution can be interpreted as due to the square of the spin-four operator $T^2 + \bar{T}^2$. Therefore, result (5.5) supports conjecture (d0) and is compatible with the stronger ones (d1) and (d2).

In [12] we also analysed the on-shell renormalization constant $Z(\tau)$ and found no terms of order τ^2. We thought this to be a good indication of the absence of both $T\bar{T}$ and $T^2 + \bar{T}^2$. We now believe that this conclusion was a little bit too hasty. First, (5.3) implies

$$Z(\tau) = \sqrt{8}a(\tau)^{1/4}b(\tau)^2 \tag{5.6}$$

with no corrections to all orders. Of course, we cannot take this as an indication that all operators are absent. Moreover, there is also a conceptual problem: $Z(\tau)$ is defined from the behaviour of the two-point function at $p = -iM(\tau)$ and thus we should consider the momentum-dependent nonlinear scaling fields as we did in [12] for the second-moment correlation length. As we shall see in the next section, no particular simplification occurs in the triangular case, and we find corrections of order τ^2 to expression (5.6). Thus, the observed cancellation is accidental and does not have any connection with the operator structure of the model.

Finally, we present an argument to make plausible the fact that the functions $\hat{F}_\pm(\tau)$ are even in τ. If the short-distance part $B_f(\tau)$ were absent, such a property would follow from the symmetry

$$(-\tau)^{-1/4}k(-\tau)^{-1/4}\chi_\pm(-\tau) = \tau^{-1/4}k(\tau)^{-1/4}\chi_\pm(\tau). \tag{5.7}$$

The interesting observation is that this symmetry property is satisfied by the large-distance expression of $G(x, y; \tau)$. Indeed, using the expressions reported above we immediately verify that

$$(-\tau)^{-1/4}k(-\tau)^{-1/4}G(x, y; -\tau) = \tau^{-1/4}k(\tau)^{-1/4}G(x, y; \tau). \tag{5.8}$$

5.2. The triangular lattice

We now repeat the same analysis on the triangular lattice. The large-distance behaviour of the two-point function along a side of the lattice was computed in [24]. Such expression was generalized in [25] where it was conjectured that the large-distance behaviour was given by the propagator of a Gaussian field on a triangular lattice, in analogy with the square-lattice expression. Therefore,

$$G(x, y; \tau) = \frac{\sqrt{3}}{8\pi^2} Z(\tau) \int_{-\pi}^{\pi} \mathrm{d}k_1 \int_{-2\pi/\sqrt{3}}^{2\pi/\sqrt{3}} \mathrm{d}k_2 \frac{\mathrm{e}^{\mathrm{i}k_1 x + \mathrm{i}k_2 y}}{\Delta_t(k) + M_t(\tau)^2} \tag{5.9}$$

where

$$\Delta_t(k) = 4 - \frac{4}{3}\cos k_1 - \frac{8}{3}\cos\frac{k_1}{2}\cos\frac{\sqrt{3}k_2}{2} \tag{5.10}$$

$$M_t(\tau)^2 = \tfrac{8}{3}\left(\cosh\tfrac{1}{2}\mu_l - 1\right)\left(\cosh\tfrac{1}{2}\mu_l + 2\right) \tag{5.11}$$

$$Z(\tau) = \tfrac{8}{3}A(\tau)^{-1/4}(k(\tau)^2 - 1)^{1/4}(A(\tau) + \sqrt{A(\tau)} + 1)^{1/2} \tag{5.12}$$

$$\mu_l(\tau) = \log A(\tau) \tag{5.13}$$

and

$$A(\tau) \equiv \left(\frac{\sqrt{1 - v + v^2} - \sqrt{v}}{\sqrt{v}(1 - v)}\right)^2. \tag{5.14}$$

The conjectured form (5.9) was checked in the high-temperature limit [25], by computing the expansion of $G(x, y; \tau)$ in powers of β to order β^{15}.

Note that, under $\tau \to -\tau$, we have

$$A(-\tau) = \frac{1}{A(\tau)} \tag{5.15}$$

and

$$M_t(-\tau)^2 = M_t(\tau)^2 \tag{5.16}$$

$$Z(-\tau)(-\tau)^{-1/4}k(-\tau)^{-1/4} = Z(\tau)\tau^{-1/4}k(\tau)^{-1/4}. \tag{5.17}$$

From the large-distance behaviour of the two-point function, we can obtain the angle-dependent correlation length $\xi(\theta)$ taken along a direction forming an angle θ with a side of the triangles. We have, in terms of the function $a(\tau)$ defined in (2.16) and (2.32),

$$\xi(\theta) = \frac{\sqrt{3}}{2a(\tau)}\left[1 + \frac{a(\tau)^4\cos 6\theta}{6480} - \frac{a(\tau)^6\cos 6\theta}{54\,432}\right.$$
$$\left. + \frac{a(\tau)^8}{55\,987\,200} + \frac{a(\tau)^8\cos 6\theta}{559\,872} - \frac{a(\tau)^8\cos 12\theta}{18\,662\,400}\right]. \tag{5.18}$$

This result provides a very strong check of conjecture (d2) presented in the introduction. Indeed, the first correction term appears only at order $a(\tau)^4$ and is proportional to $\cos 6\theta$. It is thus unambiguously related to the spin-six operator $T^3 + \bar{T}^3$. At order $a(\tau)^6$ there is also a correction term, but it is again proportional to $\cos 6\theta$ and thus it should be associated with a spin-six operator. Since no new operator appears at this order, it must be identified with an analytic correction due to the operator $T^3 + \bar{T}^3$. At order $a(\tau)^8$ a constant term and a $\cos 12\theta$ appear, but they may be due to the square of the operator $T^3 + \bar{T}^3$.

In conclusion, this calculation provides very strong evidence for the absence of $T\bar{T}$, conjecture (d1)—more precisely it proves that $\mu_{T\bar{T}} = o(\tau^4)$—and also for the conjecture (d2). Indeed, if (d1), but not (d2), were true, the spin-zero operator $Q_4^I + \bar{Q}_4^I$ would contribute to order τ^6, giving rise to an angle-independent term proportional to $a(\tau)^6$. The absence of such a term supports the validity of (d2).

Interestingly enough, this calculation allows the computation of the first analytic term in the scaling field $\mu_1(\tau)$ that is associated with $T^3 + \bar{T}^3$. Indeed, if conjecture (d2) is correct, the function $a(\tau)$ given in (2.32) coincides with the temperature scaling field at $h = 0$ up to terms of order τ^9, i.e. $\mu_t(\tau) = a(\tau) + O(\tau^9)$. Then, we write

$$\xi(\theta) = \frac{\sqrt{3}}{2}\frac{1}{\mu_t(\tau)}\left(1 + \alpha\mu_t(\tau)^4\mu_1(\tau)\cos 6\theta + O(\tau^8)\right) \tag{5.19}$$

and fix α by requiring $\mu_1(0) = 1$. Then

$$\mu_1(\tau) = 1 - \tfrac{5}{42}\tau^2 + O(\tau^4). \tag{5.20}$$

Considering now the function $Z(\tau)$, no particular simplification occurs and a correction term of order $a(\tau)^2$ appears. Explicitly

$$Z(\tau) = \frac{16}{3 \times 6^{1/4}} a(\tau)^{1/4} b(\tau)^2 \left(1 + \frac{a(\tau)^2}{18} + \cdots\right). \tag{5.21}$$

As we have already discussed in section 5.1, the presence of the quadratic term is probably related to the presence of a momentum-dependent contribution to the nonlinear scaling fields.

Finally, we note that (5.8) is also satisfied on the triangular lattice, as may be easily shown by using (5.16) and (5.17). Again, this gives a plausibility argument for the fact that the function $\hat{F}(\tau)$ appearing in (2.22) is even on the triangular lattice too.

6. Finite-size scaling at the critical point

Recently, there has been much effort in understanding the behaviour of the Ising model in a finite box or strip of size L at the critical point $h = \tau = 0$, computing the finite-size free energy f_L, energy E_L, specific heat C_L and inverse mass gap ξ_L. The results obtained are the following:

- In [26] and [7], f_L and ξ_L were computed on a strip of width L for several different lattices: it was found that these two quantities have an expansion of the form

$$L^2(f_L - f_\infty) = \sum_{n=0}^{\infty} \frac{f_n}{L^{2n}} \tag{6.1}$$

$$\frac{\xi_L}{L} = \sum_{n=0}^{\infty} \frac{s_n}{L^{2n}}. \tag{6.2}$$

  Note that in the expansion only even powers of L appear. Moreover, on a triangular lattice $f_1 = f_3 = 0$ and $s_1 = s_3 = 0$.

- Salas [9] and Izmailian and Hu [8] computed f_L, E_L, C_L for a square lattice $L \times M$ for fixed aspect ratio $\rho = M/L$, extending the results of [27, 28]. They found

$$L^2(f_L - f_\infty) = \sum_{n=0}^{\infty} \frac{f_n(\rho)}{L^{2n}} \tag{6.3}$$

$$E_L = -\sqrt{2} + \sum_{n=0}^{\infty} \frac{e_n(\rho)}{L^{2n+1}} \tag{6.4}$$

$$C_L = \frac{8}{\pi} \log L + \sqrt{2} E_L + \sum_{n=0}^{\infty} \frac{h_n(\rho)}{L^{2n}}. \tag{6.5}$$

  The specific heat has also been computed for a square lattice with Brascamp–Kunz boundary conditions in [29]. However, in this case translation invariance is lost in one direction and thus we cannot apply straightforwardly the results presented here.

In this section, we want to explain the general features of these results.

In finite volume, the general scaling expression (4.1) can be generalized by writing (see, e.g., [20, 21, 30, 31])

$$F(\tau, h; L) = f_b(\tau, h) + \frac{1}{L^2} W\left(\{u_j L^{y_j}\}\right) + \frac{1}{L^2} \log L \, \tilde{W}\left(\{u_j L^{y_j}\}\right) \tag{6.6}$$

where we assume that the bulk contribution is independent of L or, more plausibly, that it depends on L only through exponentially small corrections [20, 21], and the functions W and $\tilde{W}$ depend on all scaling fields. Equation (6.6) cannot be correct in general. Indeed, the results of [6] indicate the presence of powers of $\log |\tau|$ in the susceptibility, which imply the presence of powers of $\log L$ in (6.6). These corrections should be relevant only if we consider derivatives of the free energy with respect to h, while here we set $h = 0$ from the beginning. In this particular case, (6.6) should be correct.

If $h = 0$, the $[\sigma]$-family scaling fields do not contribute, so that (6.6) becomes

$$F(\tau, 0; L) = f_b(\tau, 0) + \frac{1}{L^2} W\left(\{\mu_j(\tau) L^{y_j}\}\right) + \frac{1}{L^2} \log L \tilde{W}\left(\{\mu_j(\tau) L^{y_j}\}\right) \tag{6.7}$$

where the scaling functions depend only on the $\mathbb{Z}_2$-even scaling fields. By using (4.5) and the fact that the RG eigenvalues y_j are even for the identity family and odd for the energy family, we obtain

$$W\left(\{\mu_j(-\tau)(-L)^{y_j}\}\right) = W\left(\{\mu_j(\tau) L^{y_j}\}\right) \tag{6.8}$$

and an analogous formula for $\tilde{W}$. Therefore, apart from the bulk contribution, even derivatives of F with respect to τ contain only even powers of L, while odd derivatives contain only odd powers of L. This explains the particular structure of the results obtained by [7–9] since

$$E_L = 2\sqrt{2} \left.\frac{\partial F}{\partial \tau}\right|_{\tau=0} \tag{6.9}$$

$$C_L = \sqrt{2} E_L + 8 \left.\frac{\partial^2 F}{\partial \tau^2}\right|_{\tau=0}. \tag{6.10}$$

In particular, (6.10) explains why the odd terms in the expansion of C_L are related to those of the energy.

For what concerns the logarithms, only C_L shows a logarithmic dependence, and only at leading order in L. This may be explained if

$$\tilde{W}\left(\{\mu_j(\tau) L^{y_j}\}\right) = \hat{W}(\mu_t(\tau) L). \tag{6.11}$$

By using the results for the specific heat at criticality and in the infinite-volume limit, we can compute the asymptotic behaviour of $\hat{W}(x)$ for $x \to 0$ and $x \to \infty$. For $x \to 0$, the results for C_L imply

$$\hat{W}(x) \approx \frac{1}{2\pi} x^2 + O(x^4) \tag{6.12}$$

while, in order to obtain the correct infinite-volume limit, we should have

$$\hat{W}(x) \approx \frac{1}{2\pi} x^2 (1 + O(x^{-2})). \tag{6.13}$$

These two results make natural the conjecture that

$$\hat{W}(x) = \frac{1}{2\pi} x^2 \tag{6.14}$$

for all x. There are several consequences of these results:

- Relation (6.11) and conjecture (c1) imply conjecture (c2), i.e. the function $\tilde{f}$ in (4.1) is a simple constant, as originally suggested by Aharony and Fisher [1]. If this is the case, the function $\mu_t(\tau)$ coincides with the function $a(\tau)$.
- If (6.14) is correct, we predict that in the expansion of $\partial^{2n} F / \partial \tau^{2n}$ at the critical point there is only one logarithmic term, with a coefficient that can be computed from the expansion of $a(\tau)$.

Let us now use (6.7) to determine the corrections to the leading behaviour. We obtain

$$L^2 f_L = L^2 f_b(0, 0) + W(\{x_j\}) \tag{6.15}$$

$$\frac{\partial F}{\partial \tau}(0) = \frac{\partial f_b}{\partial \tau}\bigg|_{\tau=h=0} + \frac{1}{L^2} \sum_{i \in [\epsilon]} L^{y_i} W_i(\{x_j\}) \tag{6.16}$$

$$\frac{\partial^2 F}{\partial \tau^2}(0) = \frac{\partial^2 f_b}{\partial \tau^2}\bigg|_{\tau=h=0} + \frac{1}{L^2} \sum_{ik \in [\epsilon]} L^{y_i+y_k} W_{ik}(\{x_j\}) + \frac{1}{L^2} \sum_{i \in [I]} \mu_{2,i} L^{y_i} W_i(\{x_j\}) + 2A \log L \tag{6.17}$$

where we write $\mu_j(\tau) = \mu_j(0) + \tau \mu_{1,j} + \frac{1}{2} \tau^2 \mu_{2,j}$, the functions W_i and W_{ik} depend only on the identity-family scaling fields through $x_j \equiv \mu_j(0) L^{y_j} = L^{y_j}$, and the constant A is defined by (2.16). We have also used the normalization conditions $\mu_{1,i} = 1$ for the energy-family fields and $\mu_j(0) = 1$ for the identity-family fields.

Let us now discuss which corrections should be expected. The important point is that here, at variance with the infinite-volume case, nonzero-spin operators can contribute to first order. Indeed, the box breaks the rotational invariance down to the lattice invariance and therefore the mean value of a lattice operator that is not rotationally invariant but has the symmetries of the lattice is nonzero. This implies that no missing term is expected on the square lattice, in agreement with the exact results. Indeed, the lowest operator is the spin-four operator $T^2 + \bar{T}^2$ that has $y = -2$ and belongs to the identity family, and is therefore able, alone, to give rise to all observed corrections.

On the triangular lattice instead simplifications are expected. Consider first the free energy f_L. The absence of the term proportional to L^{-2}, i.e. $f_1 = 0$, implies $\mu_{T\bar{T}}(0) = 0$, confirming once again conjecture (d0). The next-to-leading operator belonging to the identity family is the spin-six $T^3 + \bar{T}^3$ that has $y = -4$. Therefore, in (6.15) the $T^3 + \bar{T}^3$ gives rise to corrections of order L^{-4n}. The absence of the $1/L^6$ term requires an additional cancellation, i.e. $\mu(0)$ for the operator $Q_4^I \bar{Q}_4^I$ that has $y = -6$ and zero spin, thereby supporting our conjecture (d2). At order $1/L^8$ there appears a new operator $Q_2^I \bar{Q}_8^I + \bar{Q}_2^I Q_8^I$ that gives, together with $T^3 + \bar{T}^3$, corrections of order L^{-8n-4m} and thus indistinguishable from those of $T^3 + \bar{T}^3$. At order $1/L^{10}$, at least the spin-12 operator $T^6 + \bar{T}^6$ appears and therefore we expect all corrections of the form $L^{-10n-4m}$ to be nonvanishing.

An analogous cancellation is expected for E_L. For E_L the leading correction terms are

$$\frac{1}{L} \mu_{1,t} W_t(\{x_j\}) + \frac{1}{L^7} \mu_{1,1} W_1(\{x_j\}) + \cdots \tag{6.18}$$

where $\mu_1(\tau)$ is the scaling field of the spin-six operator $Q_6^\epsilon + \bar{Q}_6^\epsilon$ that has $y = -5$. Reasoning as before, on the basis of conjecture (d0) alone, we expect no correction of order $1/L^3$ but the presence of all other terms. Analogously in C_L the L^{-2} correction should be absent.

The results for the correlation length show the same pattern of the free energy. The fact that $s_1 = s_3 = 0$ on the triangular lattice provides additional evidence for the absence of spin-zero operators in the theory.

It is interesting to note that a similar finite-size scaling analysis was performed more than 10 years ago for the one-dimensional Ising quantum chain which belongs to the same universality class of the two-dimensional Ising model (for a discussion of their connection, see [32]). In particular, in [33] the finite-size behaviour of the free energy and of the mass spectrum of the model was obtained and then compared in [13, 17] with the predictions of perturbed CFT (see [18] for an updated review of the subject).

Remarkably enough, also in this case the contribution of the $T\bar{T}$ operator exactly disappears and the first nonzero correction is given again by the spin-four operator $T^2 + \bar{T}^2$ [13].

7. Finite-size scaling of the susceptibility at $t = 0$

In the previous section, we have discussed several thermal quantities at the critical point and verified that the observed behaviour is consistent with the RG and CFT predictions and the conjectures we have made. Here, we want to discuss the finite-size behaviour of the susceptibility on the square lattice, and we will check that the correction coefficients depend on the shape of the domain as predicted by the spin nature of the operators.

For this purpose, we study two different finite square lattices in order to verify the dependence of the corrections on the domain:

$$D_M^{(A)} = \left\{ (n_0, n_1) \in \mathbb{Z}^2, 0 \leqslant n_1, n_2 \leqslant M - 1 \right\} \tag{7.1}$$

$$D_M^{(B)} = \left\{ (n_0, n_1) \in \mathbb{Z}^2, 0 \leqslant n_1 + n_2 \leqslant 2M - 1, 0 \leqslant n_1 - n_2 \leqslant 2M - 1 \right\}. \tag{7.2}$$

In both cases the domain is a square: the first one has boundaries that are parallel to the lattice axes and size $L = M$, while the second one is rotated by $45°$ and has size $L = M\sqrt{2}$. We use periodic boundary conditions. For domain (A) such conditions are obvious, for domain (B) we identify (n_1, n_2) with $(n_1 + M, n_2 + M)$ and $(n_1 + M, n_2 - M)$.

7.1. Renormalization-group analysis

The finite-size scaling behaviour of the susceptibility can be derived easily, starting from (6.6). As we have already stated, such an expansion misses some important corrections proportional to higher powers of $\log L$. However, they should only be of interest if we analyse the asymptotic behaviour of χ for $\tau \to 0$. Here, we consider χ at the critical point and thus such corrections should vanish.

A simple computation gives at the critical point

$$\chi_L(0, 0) = \left. \frac{\partial^2 f_b}{\partial h^2} \right|_{\tau = h = 0} + \frac{1}{L^2} \sum_{k \in [I], [\epsilon]} \lambda_k(0) L^{y_k} W_k(\{x_j\}) + \frac{1}{L^2} \sum_{ik \in [\sigma]} L^{y_i + y_k} W_{ik}(\{x_j\}) \tag{7.3}$$

where the functions depend only on the identity-family scaling fields, $x_j \equiv \mu_j(0) L^{y_j} = L^{y_j}$, and we have used the normalization conditions $v_i(0) = 1$, $\mu_j(0) = 1$ for spin- and identity-family scaling fields.

Since $y_j = 2n - \frac{1}{8}$ for the $[\sigma]$-family operators and $y_j = 2n$ for the identity-family operators, where n is an integer, we have

$$\frac{1}{L^2} \sum_{ik \in [\sigma]} L^{y_i + y_k} W_{ik}(\{x_j\}) = L^{7/4} \sum_{k=0}^{\infty} \frac{c_k}{L^{2k}} \tag{7.4}$$

i.e. the corrections contain only even powers of L. On the square lattice, we do not anticipate any cancellation, i.e. we expect $c_k \neq 0$ for all k. Indeed, the leading correction is due to the operator $T^2 + \bar{T}^2$, which has $y = -2$, and thus gives rise to corrections involving all powers of L^{-2}. On the triangular lattice instead we expect $c_1 = 0$, because of conjecture (d0). All other terms are expected to be nonvanishing. Indeed, the presence of the spin-six operator $T^3 + \bar{T}^3$ generates terms L^{-4n}, while the presence of the spin-six operator $Q_6^\sigma + \bar{Q}_6^\sigma$ together with the previous one generates terms L^{-6-4n}.

Let us now consider the term that contains a sum over all identity- and energy-family operators. We expect in this case all powers of L^{-1}, i.e.

$$\frac{1}{L^2} \sum_{k \in [I], [\epsilon]} \lambda_k(0) L^{y_k} W_k(\{x_j\}) = \frac{1}{L} \sum_{k=0}^{\infty} \frac{d_k}{L^k}. \tag{7.5}$$

On the square lattice, we should have $d_1 = 0$. Indeed, the leading energy-family scaling field is associated with the temperature and gives a contribution of the form

$$\frac{1}{L^2}\lambda_t(0)LW_t(\{x_j\}) \sim \frac{1}{L}\left(a + \frac{b}{L^2} + \frac{c}{L^4} + \cdots\right) \tag{7.6}$$

and thus generates all even terms in (7.5). The odd terms in (7.5) are generated by the identity-family operators, the leading one being $T^2 + \bar{T}^2$. It gives

$$\frac{1}{L^2}\lambda_1(0)L^{-2}W_1(\{x_j\}) \sim \frac{1}{L}\left(\frac{a}{L^3} + \frac{b}{L^5} + \frac{c}{L^7} + \cdots\right) \tag{7.7}$$

and thus generates all odd terms except the first one. Hence $d_1 = 0$. Note that cancellation follows from CFT alone and does not require any additional hypothesis.

On the triangular lattice, the discussion is similar although a little more complicated. We predict $d_1 = d_2 = d_3 = d_7 = 0$. The condition $d_1 = 0$ does not require any conjecture, while $d_2 = 0$ implies the validity of conjecture (d0). Much more interesting is to check whether $d_3 = d_7 = 0$, since the vanishing of these coefficients implies $\lambda_{T\bar{T}}(0) = 0$ and $\lambda(0) = 0$ for the operator $Q_4^I \bar{Q}_4^I$. Thus, the analysis of χ on the triangular lattice would provide some additional evidence for or rule out the conjectures (d1) and (d2).

7.2. The transfer-matrix calculation

From the previous discussion, we can write on the square lattice

$$\chi_L(0,0) = L^{7/4}\left(c_0 + \frac{c_1}{L^2} + \frac{c_2}{L^4}\right) + D_0 + L^{-1}\left(d_0 + \frac{d_2}{L^2} + \frac{d_3}{L^3}\right) + O(L^{-17/4}, L^{-5}). \tag{7.8}$$

The constant D_0 is lattice and geometry independent being generated by the bulk free energy, and it is given by $B_f(0)$. Explicitly

$$D_0 = B_f(0) \approx -0.104\,133\,245\,093\,831\,026\,452\,160\,126\,860\,473\,433\,716\,236\,727\,314. \tag{7.9}$$

The other constants depend on the geometry of the system and in general are expected to be different for the two domains (A) and (B). However, this should depend on the type of operator that generates them. If a term is associated with a spin-zero operator its value should be identical in both geometries, while if it is the first contribution of a spin-four operator we expect a dependence of the form $\cos 4\theta$, where θ is the angle between the boundaries of the domain and the lattice axes. For our specific case, since $\theta = \frac{\pi}{4}$ we expect the coefficient to change sign. Therefore, we predict

$$c_0^A = c_0^B \qquad c_1^A = -c_1^B \qquad d_0^A = d_0^B. \tag{7.10}$$

Indeed, c_0 and d_0 are related to the magnetic and to the thermal scaling fields that have both spin zero. On the other hand, c_1 is related to the leading identity-family operator with $y = -2$. If the conjecture (d0) is correct, this term should be due only to the spin-four operator $T^2 + \bar{T}^2$ and thus, according to the previous discussion, it should differ by a sign in the two geometries.

In the following, we shall test the predictions (7.10). For this purpose, it is interesting to note that the constants d_0^A and d_2^A can be predicted by using the results of [8, 9, 27]. Indeed,

$$\lambda_t(0)W_t(\{x_j\}) = d_0 + \frac{d_2}{L^2} + O(L^{-3}) \tag{7.11}$$

since the leading irrelevant operator contributing to (7.5) has $y = -2$. Now $\lambda_t(0)$ is given in (4.23), while the leading contributions to the left-hand side can be derived from the energy at the critical point, since

$$E_L = 2\sqrt{2}\frac{\partial F}{\partial \tau}(0) = 2\sqrt{2}\frac{\partial f_b}{\partial \tau}\bigg|_{\tau=h=0} + \frac{2\sqrt{2}}{L}W_t(\{x_j\}) + O(L^{-5}). \tag{7.12}$$

Table 1. Numerical estimate of the magnetic susceptibility for geometry (A). In the second column, we give the results obtained by differentiation of the free energy and in the third column those obtained by summing the time-slice two-point correlation function.

L	χ	χ
4	12.181 742 537 099	12.181 742 537 098 76
5	18.092 431 830 874	18.092 431 830 873 97
6	24.959 397 280 867	24.959 397 280 866 72
7	32.740 662 899 119	32.740 662 899 118 72
8	41.402 340 799 629	41.402 340 799 631 27
9	50.915 891 978 613	50.915 891 978 613 91
10	61.256 768 274 856	61.256 768 274 858 05
11	72.403 538 830 976	72.403 538 830 975 85
12	84.337 262 930 730	84.337 262 930 726 81
13	97.041 023 059 667	97.041 023 059 664 30
14	110.499 570 854 40	110.499 570 854 393 3
15	124.699 054 324 25	124.699 054 324 247 8
16	139.626 804 325 71	139.626 804 325 709 1
17	155.271 164 846 86	155.271 164 846 852 3

For geometry (A), using the results of [8, 9, 27], we have

$$W_t(\{x_j\}) = w_{t1} + \frac{1}{L^2} w_{t2} + O(L^{-4}) \tag{7.13}$$

where

$$w_{t1} = -\frac{1}{\sqrt{2}} \frac{\theta_2(0)\theta_3(0)\theta_4(0)}{\theta_2(0) + \theta_3(0) + \theta_4(0)} \approx -0.220\,065\,581\,798\,270\,538\,286\,514\,481\,651 \tag{7.14}$$

$$w_{t2} = \frac{\pi^3}{96\sqrt{2}} \frac{\theta_2(0)\theta_3(0)\theta_4(0)\left[\theta_2(0)^9 + \theta_3(0)^9 + \theta_4(0)^9\right]}{[\theta_2(0) + \theta_3(0) + \theta_4(0)]^2}$$

$$\approx 0.073\,073\,526\,812\,330\,794\,515\,803\,384\,757 \tag{7.15}$$

so that

$$d_0^A \approx 0.022\,366\,948\,354\,353\,361\,434\,648\,349\,198 \tag{7.16}$$

$$d_2^A \approx -0.007\,427\,021\,467\,537\,379\,563\,283\,082\,599. \tag{7.17}$$

Note that this calculation relies only on the RG and on the CFT classification of the operators, but does not make use of any of the additional conjectures.

In order to check equations (7.8) and (7.10), we performed a transfer-matrix (TM) calculation of the susceptibility. Note that in general it is more difficult to perform a TM calculation in the case where both sizes of the lattice are finite than in the case where one of them is infinite, since one has to keep into account all the eigenvalues of the TM.

7.2.1. Numerical results. Let us see in detail the two cases that we studied:

- *Geometry (A).* In this case, we computed χ on lattices of sizes up to $L = 17$. In order to test our methods, we evaluated the susceptibility in two ways, by direct differentiation of the free energy and by using the fluctuation–dissipation theorem, i.e. by summing the two-point function. The results are reported in table 1. By comparing the two columns, one can estimate the size of the systematic errors.

Table 2. Numerical result for the inverse of the magnetic susceptibility for geometry (B).

M	$1/\chi$
2	0.149 678 741 567 431
3	0.073 301 790 137 056
4	0.044 241 139 068 172
5	0.029 917 172 878 427
6	0.021 735 601 983 740
7	0.016 591 966 498 537
8	0.013 132 015 183 494
9	0.010 684 547 791 392
10	0.008 884 576 737 074
11	0.007 519 096 948 920
12	0.006 456 674 647 995

- *Geometry (B)*. In order to study geometry (B) we used the following trick. As a first step, we performed a decimation of the lattice, i.e. every second spin was integrated. In this way, the number of spins is reduced by half. The price one has to pay is that the Hamiltonian becomes more complicated and contains, in addition to the nearest-neighbour interaction, a next-to-nearest neighbour and a four-point interaction. In the presence of an external field also a three-point term arises.

  However, now the axes of the decimated lattice are parallel to the axes of the torus. Also, the new Hamiltonian only couples neighbouring time slices. Therefore, we can apply the same TM methods used in geometry (A).

  Our numerical results are given in table 2. We computed the magnetic susceptibility by differentiation of the free energy. The largest lattice has $M = 12$, which corresponds to $L = 16.98$, and is thus completely equivalent to the largest lattice used in geometry (A).

7.2.2. Analysis of the data. We will now use the TM data to check the theoretical predictions. We expect that the error induced by the error on χ given in tables 1 and 2 is small compared to that due to the neglected higher-order corrections in (7.8). Therefore, instead of performing a fit, we considered as many data points as the number of free parameters of the ansatz, and then required the ansatz to be exact for them. This gives a system of equations that is then solved for the free parameters. We always used consecutive values of L, i.e. $L_1 = L$, $L_2 = L - 1, \ldots, L_n = L - n + 1$, where n is the number of free parameters. Errors were estimated from the variation of the results with the lattice size and by comparison of different ansätze.

As a preliminary test, we checked that $y = -2$ for the leading correction to scaling. For this purpose, we studied the ansatz

$$\chi_L(0, 0) = L^{7/4} \left(c_0 + c_1 L^y\right) + D_0 \tag{7.18}$$

with c_0, c_1, and y as free parameters. The results are summarized in table 3. For both geometries, the numerical result for y approaches -2 as L increases. For our largest lattice sizes, the deviation from -2 is about 1%. In the following analysis, we shall assume $y = -2$.

Next we analysed the data with (7.8). For geometry (A), by using the known values of D_0, d_0 and d_2, we found

$$c_0^A = 1.091\,950\,56(4) \tag{7.19}$$

$$c_1^A = -0.079\,14(5) \tag{7.20}$$

Table 3. Numerical results from the ansatz (7.18) in geometries (A) and (B).

	c_0	c_1	y
	Geometry (A)		
L			
12	1.091 9299	−0.0964	−2.102
13	1.091 9370	−0.0915	−2.076
14	1.091 9414	−0.0881	−2.057
15	1.091 9441	−0.0857	−2.044
16	1.091 9460	−0.0838	−2.034
17	1.091 9472	−0.0823	−2.026
	Geometry (B)		
M			
8	1.091 9297	0.0689	−1.922
9	1.091 9388	0.0720	−1.946
10	1.091 9435	0.0743	−1.962
11	1.091 9461	0.0761	−1.973
12	1.091 9477	0.0775	−1.982

where the quoted uncertainties were obtained by comparing the results of the ansatz (7.8) with those obtained by adding c_3 as a free parameter.

For geometry (B), by using the known value of D_0, we obtain

$$c_0^B = 1.091\,950\,4(2) \tag{7.21}$$

$$c_1^B = 0.0794(4) \tag{7.22}$$

$$d_0^B = 0.019(5). \tag{7.23}$$

Our predictions (7.10) appear to be very well satisfied. Moreover, our result for c_0 is in good agreement with, although much more precise than, the estimate[12] of [22], $c_0 = 1.092\,10(11)$.

If we assume $d_0^B = d_0^A$ and use (7.16), we obtain the more precise estimate

$$c_0^B = 1.091\,950\,6(2) \tag{7.24}$$

$$c_1^B = 0.0790(2) \tag{7.25}$$

where the error was obtained by comparing the results with and without the parameter d_2.

From the above analysis we see that, within the errors, the coefficients of the $1/L^2$ correction are equal in magnitude and opposite in sign for the two geometries. Since the two lattices are rotated by $\pi/4$ this implies that this correction *is completely due to the spin-four operator $T^2 + \bar{T}^2$ and that the scalar operator $T\bar{T}$ is absent*, in agreement with conjecture (d0).

8. Concluding remarks and open issues

In this paper, we have discussed the presently available results for the corrections to scaling in the two-dimensional Ising model. We have shown that all results are in agreement with the RG and CFT predictions. The only missing point here is a complete analysis of the RG resonances and consequently an extension of the scaling forms (4.1) and (6.6) to take into account the logarithmic structure found in [6]. We have also shown that the existence of an exact symmetry

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[12] We report here the result of their fit with $\Delta = 7/4$, since this is the correct theoretical behaviour.

in the lattice models that map the high-temperature phase onto the low-temperature one plays a very important role and explains the symmetry properties of the results.

However, the lattice Ising model shows also features that are *not* predicted by CFT and RG and that can be explained if some additional conjectures are made. A list of these is reported in section 4.1. Let us summarize the evidence we have:

- *Conjectures (a) and (b)*. They allow us to explain the symmetry properties under $\tau \to -\tau$ of the free energy and of its derivatives for $h = 0$. Further evidence may be obtained by analysing χ on the triangular lattice and checking whether the functions $\hat{F}_{\pm}(\tau)$ are even in τ.

- *Conjecture (c1)*. The functions $\tilde{f}_{\pm}$ do not depend on the $[\sigma]$-family fields. This is supported by the exact known results for $F(\tau, 0)$ and $M(\tau, 0)$ and by the results of [6]. Further evidence is obtained from the absence of a leading logarithmic correction in higher-point correlation functions [10, 11].

- *Conjecture (c2)*. The functions $\tilde{f}_{\pm}$ are constants (this is the original conjecture of [1]). The independence of $\tilde{f}_{\pm}$ from the $\mathbb{Z}_2$-even scaling fields is supported by the finite-size results of [8, 9] discussed in section 6. The conjecture follows from this observation and conjecture (c1). Conjecture (c2), together with the conjectured formula (6.14) can be checked further by computing the logarithmic term(s) in $\partial^n F/\partial \tau^n$ at the critical point for $n > 2$.

- *Conjecture (d0)*. The nonlinear scaling field of $T\bar{T}$ vanishes at the critical point. On the square lattice we have ample evidence in favour of (d0), which is the only conjecture needed to explain the existing results. Indeed, it is supported by

  (1) the infinite-volume results of [6];
  (2) the behaviour of $\xi(\theta)$ discussed in section 5.1;
  (3) the dependence of χ at the critical point from the boundary conditions, see section 7;
  (4) the behaviour of the two-point function at the critical point, see [34];
  (5) the behaviour of the free energy on the critical isotherm, see [3];

  Moreover, all triangular-lattice results are compatible with it. For these reasons, we believe it is more than a conjecture and it is essentially proved. It is interesting to note that a similar cancellation is observed in the finite-size scaling behaviour of the free energy and of the mass spectrum in the one-dimensional Ising quantum chain, see [13].

- *Conjecture (d1)*. The operator $T\bar{T}$ is decoupled. We have evidence for the validity of this conjecture in the triangular-lattice Ising model. The analysis of the correlation length $\xi(\theta)$ on the triangular lattice shows that $\mu_{T\bar{T}}(0)$ vanishes at least up to terms of order $O(\tau^6)$. There are several calculations that should be feasible and would add further support to the validity of (d1) on the triangular lattice:

  (1) The extension of the results of [6] to the triangular lattice.
  (2) The study of the dependence on the boundary conditions of the observables studied in section 6 at the critical point on the triangular lattice. This would unambiguously identify the spin of the leading irrelevant operator.
  (3) The study of χ at the critical point on a triangular lattice. It is particularly important to verify whether d_3, cf (7.5), vanishes or not. If it does, it provides the only available evidence for $\lambda_{T\bar{T}}(0) = 0$, and thus it would strengthen the conjecture.

- *Conjecture (d2)*. Only nonzero-spin operators are present. We have evidence for this conjecture on the triangular lattice. The absence of spin-zero operators besides $T\bar{T}$ is based on the results of sections 5.2 and 6 where we showed that the existing exact results require $\mu(0) = 0$ for the spin-zero identity-family operator $Q_4^I \bar{Q}_4^I$ with $y = -6$. The studies (1) and (2) mentioned at the previous point would further check the conjecture. In

particular, they can verify whether $v(0) = 0$ for the spin-zero $[\sigma]$-family operator $Q_3^\sigma \bar{Q}_3^\sigma$ with $y = -4 - \frac{1}{8}$.

Of course, as they stand, these conjectures are just '*ad hoc*' prescriptions, whose only merit is that of providing an economical way to explain all existing results. It would be very important to understand if there is some symmetry argument that could explain them.

There remain several open questions. First of all, one may ask whether these conjectures apply to the nearest-neighbour Ising model on any regular lattice or whether some of them depend on the lattice structure. Another important question is how important the nearest-neighbour condition is: do some of these conjectures apply also to the Ising model with extended interactions? Finally, one may ask whether these cancellations are also observed in other models. Concerning this last question, we should mention the results of [35] for the three-state Potts quantum chain, which were compared with the CFT predictions in [13]. Again, the $T\bar{T}$ contribution turns out to be compatible with zero. However, at variance with the Ising case, there is, at next-to-leading order, a clear signature of a finite-size correction due to a *scalar* irrelevant operator. Even if the Potts case is slightly different from the Ising one, since this irrelevant operator is actually a *primary* operator (more precisely is the one with conformal weight $h = \frac{7}{5}$), this result indicates that our conjecture (d2), if true, is specific for the Ising model and could be somehow related to the fact that the model is soluble on the lattice. On the other hand, the vanishing of the correction due to the $T\bar{T}$ operator seems to be a more general phenomenon. In order to understand the validity of (d0), it would be interesting to extend these analyses to the generic q-state Potts model or to other specific values of q (for instance, to percolation).

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